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On a non-homogeneous cylindrically aelotropic, magnetostrictive rotating long cylinder

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This article presents an analytic solution of the stresses and displacements in a long rotating cylinder of nonhomogeneous cylindrically aelotropic, magnetostrictive material of variable density.

1. Introduction

The constitutive relations of a continuous, isotropic, magnetostrictive medium have been established by Lewis (1962) from the standpoint of the theory of classical mechanics of continuous media. Thus it has been made possible to study the interaction of elastic field with magnetic field, when magnetic fields (small) are accommodated in classical problems of elasticity. The elastic problems on magnetostrictive material have been discussed in the recent papers of Sinha (1962, 1967) and of Giri (1963). In our present paper, the problem of an elastically non-homogeneous, cylindrically aelotropic, magnetostrictive, uniformly rotating, long thickwalled-hollow circular cylinder of variable density, where the elastic and the magnetostrictive constants vary radially according to power law, has been investigated. The cylindrical coordinates (r, θ, z) are used such that the z-axis coincides with the axis of the cylinder. The constitutive relations (Lewis 1952) have been modified in the present case. The problem is reduced to the solution of a second order differential equation in radial displacement.

2. Problem, fundamental equations and boundary conditions

Let r_0 and r_a be the inner and outer radii respectively of the hollow thick cylinder of cylindrically aelotropic, magnetostrictive material, rotating uniformly with angular velocity Ω about its axis. The cylindrical bounding surfaces are supposed to be free from radial stress. The subsequent analysis is done under the plane strain condition. In additon to these mechanical conditions, we introduce a circumferential magnetic field produced by a constant longitudinal current density J_0. Our object is to obtain the stresses and displacements, resulting from the interaction of mechanical and magnetic fields within the cylinder.

Evidently, the equilibrium of motion and the relation between the strain components and displacements do not depend on the type of material and as such they remain the same as in the case of isotropy. Thus for the axially symmetric, uniformly rotating long cylinder of the material under consideration, the radial displacement u is a function of r only, the tangential displacement v is zero and the longitudinal displacement $\omega = e_o z$ where e_o is the constant extension in the axial direction. Now we take recourse to the solution of the usual stress equations of equilibrium utilising the constitutive relations.

The expressions for the strain-components are (Sokolnikoff 1956), in view of plane strain condition and symmetric displacements,

$$\left.\begin{aligned} &S_{rr} = \frac{du}{dr}\ , S_{\theta\theta} = \frac{u}{r}\ , S_{zz} = e_o \\ &S_{\theta z} = 0\ , S_{zr} = 0\ , S_{r\theta} = 0 \end{aligned}\right\} \tag{1}$$

where S_{rr}, $S_{\theta\theta}$ and S_{zz} are are respectively strain components in radial, tangential and longitudinal directions, $S_{\theta z}$, S_{zr} and $S_{\theta r}$ are the shearing strain components.

The constitutive relations, in cylindrical coordinates, as suggested by Lewis (1962) have been modified, in the present case, into

$$\left.\begin{aligned} \sigma_{rr} &= C_{11} S_{rr} + C_{12} S_{\theta\theta} + C_{13} S_{zz} + a_{11} H_r^2 + a_{12} H_\theta^2 + a_{13} H_z^2. \\ \sigma_{\theta\theta} &= C_{21} S_{rr} + C_{22} S_{\theta\theta} + C_{23} S_{zz} + a_{21} H_r^2 + a_{22} H_\theta^2 + a_{23} H_z^2. \\ \sigma_{zz} &= C_{31} S_{rr} + C_{32} S_{\theta\theta} + C_{33} S_{zz} + a_{31} H_r^2 + a_{32} H_\theta^2 + a_{33} H_z^2. \\ \sigma_{\theta z} &= C_{44} S_{\theta z} + a_{44} H_\theta H_z \\ \sigma_{zr} &= C_{55} S_{zr} + a_{55} H_z H_r \\ \sigma_{r\theta} &= C_{66} S_{r\theta} + a_{66} H_r H_\theta. \end{aligned}\right\} \tag{2}$$

where $C_{ij} = C_{ji}$, $(i,j=1, 2, 3)$ and σ_{rr}, $\sigma_{\theta\theta}$ and σ_{zz} are radial, tangential and longitudinal stresses respectively, $\sigma_{\theta z}$, σ_{zr} and $\sigma_{r\theta}$ are the shearing stresses and C_{ij} are elastic constants. H_r, H_θ, H_z are the magnetic field components in cylindrical coordinates and a_{ij} $(i,j=1, 2, 3, 4, 5, 6)$ are the magnetostrictive constants.

Now according to our previous assumption the coefficients C_{ij} and a_{ij} in (2) are taken as functions of r and let

$$\left.\begin{aligned} &C_{ij} = \mu_{ij}\, r^n, \\ &a_{ij} = \lambda_{ij}\, r^n,\ (\mu_{ij}, \lambda_{ij}, n \text{ are constants}). \end{aligned}\right\} \tag{3}$$

Since the magnetic field originates from a longitudinal current of density J_o, we have

$$H_r = H_z = 0\ ; H_\theta(r) = \frac{J_o r}{2} \tag{4}$$

From the relations (1) – (4), we have

$$\left.\begin{aligned}
\sigma_{rr} &= r^n\left(\mu_{11}\frac{du}{dr}+\mu_{12}\frac{u}{r}+\mu_{13}e_0\right)+r^n\lambda_{12}\frac{J_0^2r^2}{4}\\
\sigma_{\theta\theta} &= r^n\left(\mu_{21}\frac{du}{dr}+\mu_{22}\frac{u}{r}+\mu_{23}e_0\right)+r^n\lambda_{22}\frac{J_0^2r^2}{4}\\
\sigma_{zz} &= r^n\left(\mu_{31}\frac{du}{dr}+\mu_{32}\frac{u}{r}+\mu_{33}e_0\right)+r^n\lambda_{32}\frac{J_0^2r^2}{4}\\
\sigma_{\theta z} &= 0.\\
\sigma_{zr} &= 0 = \sigma_{r\theta}
\end{aligned}\right\} \quad \ldots(5)$$

where

$\mu_{ij} = \mu_{ji}$, $(i, j = 1, 2, 3)$.

The equilibrium equations in cylindrical coordinates are (Sokolnikoff 1956),

$$\left.\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial_\theta}+\frac{\partial\sigma_{rz}}{\partial_z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+F_r=0\\
&\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial_\theta}+\frac{\partial\sigma_{\theta z}}{\partial_z}+\frac{2\sigma_{r\theta}}{r}+F_\theta=0\\
&\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial_{\theta z}}{\partial_\theta}+\frac{\partial\sigma_{zz}}{\partial_z}+\frac{\partial_{rz}}{r}+F_z=0
\end{aligned}\right\} \quad \ldots(6)$$

where F_r, F_θ and F_z are the radial, tangential and longitudinal components of the body force F per unit volume respectively.

In the case of uniformly rotating cylinder the only body force is the centrifugal force in the radial direction and this is evidently $\rho\Omega^2 r$ where ρ is the density of the material and Ω the angular velocity of the cylinder in radians per second. But conforming to our assumption ρ is taken as

$$\rho = \rho_0\left(\frac{r}{r_0}\right)^2 \text{ where } \rho_0 \text{ is the density at } r = r_0.$$

The first equation of (6) reduces to

$$\frac{d\sigma_{rr}}{dr}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\rho_0\left(\frac{r}{r_0}\right)^2\Omega^2 r=0 \qquad \ldots(7)$$

The last two equilibrium equations are identically satisfied.

Eliminating σ_{rr} and $\sigma_{\theta\theta}$ from (5) and (7), the differential equation in radial displacement is obtained as

$$r^2\frac{d^2u}{dr^2}+(n+1)r\frac{du}{dr}+\partial r=\phi e_0 r-\psi J_0^2 r^3-\beta r^{\delta-n}. \qquad \ldots(8)$$

where

$$\partial = \frac{n\mu_{12}-\mu_{22}}{\mu_{11}}$$

$$\phi = \frac{\mu_{23}-(n+1)\,\mu_{13}}{\mu_{11}}$$

$$\psi = \frac{(n+2)\,\lambda_{12}+(\lambda_{12}-\lambda_{22})}{4\mu_{11}}$$

$$\beta = \frac{\rho_o\Omega^2}{\mu_{11}r_o^{\,2}}$$

The boundary conditions are $(\sigma_{rr})_{r=r_o} = (\sigma_{rr})_{r=r_a} = 0$

and
$$\int_{r_o}^{r^a} \sigma_{zz} r\,dr = 0.$$

3. Solution of the problem

Solving (8), we find

$$u = Ar^{\alpha_1}+Br^{\alpha_2}+\alpha_3 e_o r-\alpha_4 J_o^{\,2} r^3+\alpha_5 r^{5-n}. \qquad \ldots(9)$$

where

$$\alpha_1 = \frac{-n+\sqrt{n^2-4\delta}}{2}.$$

$$\alpha_2 = \frac{-n-\sqrt{n^2-4\delta}}{2}.$$

$$\alpha_3 = \frac{\phi}{(n+\delta+1)}.$$

$$\alpha_4 = \frac{\psi}{(3n+\delta+9)}.$$

$$\alpha_5 = \frac{\beta}{(5n-\delta-25)}.$$

and A, B are constants of integration to be determined from the boundary conditions stated earlier.

The stress components in (5) thereby become

$$\sigma_{rr} = AA_1 r^{n+\alpha_1-1}+BB_1 r^{n+\alpha_2-1}+C_1 e_o r^n-D_1 r^{n+2}+E_1 r^4+r^n\lambda_{12}H_\theta^2(r).$$

$$\sigma_{\theta\theta} = AA_2 r^{n+\alpha_1-1}+BB_2 r^{n+\alpha_2-1}+C_2 e_o r^n-D_2 r^{n+2}+E_2 r^4+r^n\lambda_{22}H_\theta^2(r).$$

$$\sigma_{zz} = AA_3 r^{n+\alpha_1-1}+BB_3 r^{n+\alpha_2-1}+C_3 e_o r^n-D_3 r^{n+2}+E_3 r^4+r^n\lambda_{32}H_\theta^2(r). \qquad \ldots(10)$$

where

$$A_i = \alpha_1\mu_{i1}+\mu_{i2}$$
$$B_i = \alpha_2\mu_{i1}+\mu_{i2}$$
$$C_i = \alpha_3(\mu_{i1}+\mu_{i2})+\mu_{i3}$$
$$D_i = (3\mu_{i1}+\mu_{i2})J_o^{2}\alpha_4$$
$$E_i = \{(5-n)\mu_{i1}+\mu_{i2}\}\alpha_5$$
$$(i = 1, 2, 3)$$

From the boundary condition

$$(\sigma_{rr})_{r=r_o} = (\sigma_{rr})_{r=r_a} = 0,$$

we have

$$AA_1r_o^{\,n+\alpha_1-1}+BB_1r_o^{\,n+\alpha_2-1}+c_1e_or_o^{\,n}-D_1r_o^{\,n+2}+E_1r_o^{\,4}+\lambda_{12}r_o^{\,n}H_{\ 2}(r_o)=0$$

$$AA_1r_a^{\,n+\alpha_1-1}+BB_1r_a^{\,n+\alpha_2-1}+C_1e_or_a^{\,n}-D_1r_a^{\,n+2}+E_1r_a^{\,4}$$
$$+\lambda_{12}r_a^{\,n}H_\theta^{\,2}(r_a) = 0 \qquad \ldots(11)$$

Now from (11) we have

$$A=\frac{C_1e_o\begin{vmatrix} r_a^{\,n+\alpha_2-1} & r_a^{\,n} \\ r_o^{\,n+\alpha_2-1} & r_o^{\,n}\end{vmatrix}-D_1\begin{vmatrix} r_a^{\,n+\alpha_2-1} & r_a^{\,n+2} \\ r_o^{\,n+\alpha_2-1} & r_o^{\,n+2}\end{vmatrix}+E_1\begin{vmatrix} r_a^{\,n+\alpha_2-1} & r_a^{\,4} \\ r_o^{\,n+\alpha_2-1} & r_o^{\,4}\end{vmatrix}+\lambda_{12}\begin{vmatrix} r_a^{\,n+\alpha_2-1} & r_aH_\theta^{\,4}(r_a) \\ r_o^{\,n+\alpha_2-1} & r_o^{\,n}H_\theta^{\,4}(r_o)\end{vmatrix}}{A_1\begin{vmatrix} r_a^{\,n+\alpha_2-1} & r_a^{\,n+\alpha_2-1} \\ r_o^{\,n+\alpha_2-1} & r_o^{\,n+\alpha_2-1}\end{vmatrix}}$$

$$B=\frac{C_1e_o\begin{vmatrix} r_a^{\,n+\alpha_1-1} & r_a^{\,n} \\ r_o^{\,n+\alpha_1-1} & r_o^{\,n}\end{vmatrix}-D_1\begin{vmatrix} r_a^{\,n+\alpha_1-1} & r_a^{\,n+2} \\ r_o^{\,n+\alpha_1-2} & r_o^{\,n+1}\end{vmatrix}+E_1\begin{vmatrix} r_a^{\,n+\alpha_1-1} & r_a^{\,4} \\ r_o^{\,n+\alpha_1-1} & r_o^{\,4}\end{vmatrix}+\lambda_{12}\begin{vmatrix} r_a^{\,n+\alpha_1-2} & r_a^{\,n}H_\theta^{\,2}(r_a) \\ r_o^{\,n+\alpha_1-1} & r_o^{\,n}H_{\ 2}(r_o)\end{vmatrix}}{B_1\begin{vmatrix} r_a^{\,n+\alpha_1-1} \\ r_o^{\,n+\alpha_1-1}\end{vmatrix}\begin{vmatrix} r_a^{\,n+\alpha_2-1} \\ r_o^{\,n+\alpha_2-1}\end{vmatrix}}$$

As in the case of isotropy, the traction σ_{zz} at the ends of the cylinder cannot be made to vanish, but can be so adjusted that they have no statical resultant.

'hus $\displaystyle\int_{r_o}^{r_a} \sigma_{zz}\, rdr = 0$

From the last equation of (10), we have therefore,

$$\frac{AA_3\left(r_a^{\,n+\alpha_1+1} - r_0^{\,n+\alpha_1+1}\right)}{(n+\alpha_1+1)} + \frac{BB_3\left(r_a^{\,n+\alpha_2+1} - r_0^{\,n+\alpha_2+1}\right)}{(n+\alpha_2+1)}$$

$$+ \frac{C_3e_0\left(r_a^{\,n+2} - r_0^{\,n+2}\right)}{(n+2)} - \frac{D_3\left(r_a^{\,n+4} - r_0^{\,n+4}\right)}{(n+4)}$$

$$+ \frac{E_3\left(r_a^{\,6} - r_0^{\,6}\right)}{6} + \frac{\lambda_{32}J_0^{\,2}\left(r_a^{\,n+4} - r_0^{\,n+4}\right)}{4(n+4)} = 0. \quad \ldots(12)$$

From (12) the axial extension e_0 can be determined with the help of the values of A and B. Now A and B completely determined. Thus the radical stress σ_{rr} can be determined.

The hoop-stresses at the inner and the outer surfaces are determined from the second relation of (10) on substituting r_0 and r_a respectively for r. The longitudinal or axial stress σ_{zz} and the radial displacement u are given by the last equation of (10) and (9) respectively. And when e_0 is known, the longitudinal displacement ω is known.

If now the longitudinal displacement is taken to be zero (as in the case when the ends of the cylinder are placed between two fixed planes), $e_0=0$ and the results for this case are obtained form the corresponding equations already deduced.

The the total axial pull across a normal section of the cylinder is

$$2\pi\int_{r_0}^{r_a} \sigma_{zz} r\,dr = 2\pi\left[\frac{AA_3}{(n+\alpha_1+1)}\left(r_a^{\,n+\alpha_1+1} - r_0^{\,n+\alpha_1+1}\right)\right.$$

$$+\frac{BB_3}{(n+\alpha_2+1)}\left(r_a^{\,n+\alpha_2+1} - r_0^{\,n+\alpha_2+1}\right) - \frac{D_3}{n+4}\left(r_a^{\,n+4} - r_0^{\,n+4}\right)$$

$$\left.+ \frac{E_3}{6}\left(r_a^{\,6} - r_0^{\,6}\right) + \frac{\lambda_{32}J_0^{\,2}}{4(n+4)}\left(r_a^{\,n+4} - r_0^{\,n+4}\right)\right] \quad \ldots(13)$$

Therefore the mean axial stress P is

$$2\left[\frac{AA_3}{(n+\alpha_1+1)}\left(\frac{r_a^{\,n+\alpha_1+1}-r_o^{\,n+\alpha_1+1}}{r_a^2-r_o^2}\right)\right.$$

$$+\frac{BB_3}{(n+\alpha_2+1)}\left(\frac{r_a^{\,n+\alpha_2+1}-r_o^{\,n+\alpha_2+1}}{r_a^2-r_o^2}\right)-\frac{D_3}{n+4}\left(\frac{r_a^{\,n+4}-r_o^{\,n+4}}{r^2-r_o^2}\right)$$

$$\left.+\frac{E_3}{6}\left(r_a^4+r_a^2r_o^2+r_o^4\right)+\frac{\lambda_{32}J_o^2}{4(n+4)}\left(\frac{r_a^{\,n+4}-r_o^{\,n+4}}{r_a^2-r_o^2}\right)\right] \qquad \ldots(14)$$

The above non-zero axial force must be nullified so that the longitudinal strain S_{zz} may be zero as required under the hypothesis in this case. Thus a uniform axial compression given by (14) is to be superposed. Clearly this will not affect the radial stress σ_{rr} and the tangential stress $\sigma_{\theta\theta}$. But then the longitudinal stress becomes

$$AA_3\left[r^{n+\alpha_1-1}-\frac{2}{(n+\alpha_1+1)}\cdot\frac{\left(r_a^{\,n+\alpha_1+1}-r^{\,n+\alpha_1+1}\right)}{(r_a^2-r_o^2)}\right]$$

$$+BB_3\left[r^{n+\alpha_2+1}-\frac{2}{(n+\alpha_2+1)}\cdot\frac{\left(r_a^{\,n+\alpha_2+1}-r_o^{\,n+\alpha_2+1}\right)}{(r_a^2-r_o^2)}\right]$$

$$-D_3\left[r^{n+2}-\frac{2}{n+4}\cdot\frac{\left(r_a^{\,n+4}-r_o^{\,n+4}\right)}{(r^{a2}-r_a^2)}\right]$$

$$+E_3\left[r^4-\frac{1}{3}\left(r_a+r_a^2r_o^2+r_o^4\right.\right]$$

$$+\frac{\lambda_{32}J_o^2}{4}\left[r^{n+2}-\frac{2}{(n+4)}\cdot\frac{\left(r_a^{\,n+4}-r_o^{\,n+4}\right)}{(r_a^2-r_o^2)}\right].$$

For homogenity, $n=0$ and in the absence of magnetic field, the results agree with some standard results in purely elastic case.

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